The local Miyawaki liftings and the Gan-Gross-Prasad conjecture (Representation theory and various problems in algebra, analysis, and geometry)

Atobe, Hiraku

数理解析研究所講究録 第2103巻 2019年 122-125

The local Miyawaki liftings and the Gan–Gross–Prasad conjecture

Hiraku Atobe
Graduate School of Mathematical Sciences, The University of Tokyo

Abstract

The Gan–Gross–Prasad conjecture for the Fourier–Jacobi case is a local analogous problem for the Fourier–Jacobi expansion and the theta expansion of modular forms. The Miyawaki lifting, which was constructed by Ikeda, is a lifting similar to the theta lifting. In this talk, using local Miyawaki liftings, we give a new example of the Gan–Gross–Prasad conjecture for a non-generic case.

1 The local Gan–Gross–Prasad conjecture for Fourier–Jacobi case

Fix a finite extension F of the p-adic field $\mathbb{Q}_p$. Let

- $\mathrm{Sp}_r(F)$ be the symplectic group of rank r given by

$$\mathrm{Sp}_r(F) = \left\{ g \in \mathrm{GL}_{2r}(F) \;\middle|\; {}^t g \begin{pmatrix} 0 & -\mathbf{1}_r \\ \mathbf{1}_r & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -\mathbf{1}_r \\ \mathbf{1}_r & 0 \end{pmatrix} \right\};$$

- $\widetilde{\mathrm{Sp}}_r(F)$ be the metaplectic double cover of $\mathrm{Sp}_r(F)$, which is identified with $\mathrm{Sp}_r(F) \times \{\pm 1\}$ as sets;
- $V_{r-1}(F) \subset \mathrm{Sp}_r(F)$ be a Heisenberg group given by

$$V_{r-1}(F) = \left\{ \mathbf{v}(x,y,z) = \left(\begin{array}{cc|cc} 1 & x & z & y \\ 0 & \mathbf{1}_{r-1} & {}^t y & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -{}^t x & \mathbf{1}_{r-1} \end{array} \right) \;\middle|\; x, y \in F^{r-1},\ z \in F \right\}.$$

We identify $\mathrm{Sp}_{r-1}(F)$ as a subgroup of $\mathrm{Sp}_r(F)$ by the embedding

$$\mathrm{Sp}_{r-1}(F) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ \hline 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{array} \right) \in \mathrm{Sp}_r(F).$$

Let $J_{r-1}(F) = \mathrm{Sp}_{r-1}(F) \ltimes V_{r-1}(F) \subset \mathrm{Sp}_r(F)$ be a Jacobi group, and $\widetilde{J}_{r-1}(F) = \widetilde{\mathrm{Sp}}_{r-1}(F) \ltimes V_{r-1}(F) \subset \widetilde{\mathrm{Sp}}_r(F)$ be its double cover.

Fix a non-trivial additive character ψ of F. For $\xi \in F^\times$, we set $\psi_\xi(x) = \psi(\xi x)$ for $x \in F$. The Weil representation of $\widetilde{J}_{r-1}(F)$ whose central character is ψ_ξ is denoted by $\omega_{\psi_\xi}^{(r-1)}$. By the restriction, we may also regard $\omega_{\psi_\xi}^{(r-1)}$ as a smooth representation of $\widetilde{\mathrm{Sp}}_{r-1}(F)$.

For irreducible smooth representations π_1 of $\widetilde{\mathrm{Sp}}_r(F)$ and π_2 of $\widetilde{\mathrm{Sp}}_{r-1}(F)$, set

$$d_{r,r-1,\xi}(\pi_1,\pi_2) = \dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{J}_{r-1}(F)}(\pi_1 \otimes \pi_2 \otimes \overline{\omega_{\psi_\xi}^{(r-1)}}, \mathbb{C}).$$

Similarly, for two irreducible smooth representations π_1' and π_2' of $\widetilde{\mathrm{Sp}}_n(F)$, set

$$d_{n,n,\xi}(\pi_1',\pi_2') = \dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{\mathrm{Sp}}_n(F)}(\pi_1' \otimes \pi_2' \otimes \overline{\omega_{\psi_\xi}^{(n)}}, \mathbb{C}).$$

Theorem 1.1 ([4], [3]). *For any π_1, π_2, π_1', and π_2' as above,*

$$d_{r,r-1,\xi}(\pi_1,\pi_2) \le 1, \quad d_{n,n,\xi}(\pi_1',\pi_2') \le 1.$$

The local Gan–Gross–Prasad conjecture for the Fourier–Jacobi case describes these dimensions for the tempered case.

Theorem 1.2 ([3], [1]). *When π_1, π_2, π_1', and π_2' are tempered, there exist explicit descriptions for $d_{r,r-1,\xi}(\pi_1,\pi_2)$ and $d_{n,n,\xi}(\pi_1',\pi_2')$ in terms of their L-parameters.*

The purpose of this article is to give a new example for a non-tempered case.

Let $(\cdot,\cdot)_F$ be the quadratic Hilbert symbol of F, and $\chi_\xi = (\cdot,\xi)_F$ be the quadratic character of $F^\times$ associated to $\xi \in F^\times$. A double cover of $\mathrm{GL}_k(F)$ is defined by $\widetilde{\mathrm{GL}}_k(F) = \mathrm{GL}_k(F) \times \{\pm 1\}$ as a set, and its group law is given by

$$(a_1,\epsilon_1)\cdot(a_2,\epsilon_2) = (a_1a_2, \epsilon_1\epsilon_2(\det a_1, \det a_2)_F).$$

Then there exists a genuine character $\chi_{-1}^{1/2}$ of $\widetilde{\mathrm{GL}}_1(F)$, depending on ψ, such that $\chi_{-1}^{1/2}(a,\epsilon)^2 = \chi_{-1}(a)$ for $(a,\epsilon) \in \widetilde{\mathrm{GL}}_1(F)$.

For a unitary character of $F^\times$ and an irreducible smooth representation of $\widetilde{\mathrm{Sp}}_r(F)$, we denote by $\mu \circ \det_k \rtimes \pi$ the space of locally constant function $f\colon \widetilde{\mathrm{Sp}}_{r+k}(F) \to \pi$ such that

$$f\left(\left(\left(\begin{array}{cc|cc} a & * & * & * \\ 0 & A & * & B \\ \hline 0 & 0 & {}^t a^{-1} & 0 \\ 0 & C & * & D \end{array}\right), \zeta\right) g\right) = \chi_{-1}^{1/2}(\det a, \zeta)^\delta \mu(\det a) |\det a|^{r+\frac{k+1}{2}} \pi\begin{pmatrix} A & B \\ C & D \end{pmatrix} f(g)$$

for $a \in \mathrm{GL}_k(F)$, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_r(F)$, $\zeta \in \{\pm 1\}$ and $g \in \widetilde{\mathrm{Sp}}_{r+k}(F)$, where $\delta = 1$ if π is genuine, and $\delta = 0$ otherwise.

The main result is given as follows:

Theorem 1.3 ([2, Theorems 1.8(2), Theorem C.5]). *Fix positive integers $n \ge r$. Let μ be a unitary character of $F^\times$, and π_1 and π_2 be irreducible tempered representations of $\widetilde{\mathrm{Sp}}_r(F)$ and $\widetilde{\mathrm{Sp}}_{r-1}(F)$ on which $\{\pm 1\}$ acts by $(\pm 1)^{n+r}$ and $(\pm 1)^{n+r-1}$, respectively.*

1. *The induced representation $\mu \circ \det_{n-r} \rtimes \pi_1$ is irreducible.*

2. *Assume that π_1 is discrete series, or $r \le n \le r+1$, or $n > 2r$. Then we have*

$$d_{n,n,\xi}(\mu\chi_{-1}^{n+r-1} \circ \det_{n-r} \rtimes \pi_1, \mu\chi_\xi \circ \det_{n-r+1} \rtimes \pi_2) = d_{r,r-1,\xi}(\pi_1, \pi_2).$$

In the proof of (1), we compute Jacquet modules of several induced representations using *the geometric lemma*. For the proof of (2), we use *seesaw identities for local Miyawaki liftings*.

2 The local Miyawaki liftings

Fix a unitary character μ of $F^\times$. For two positive integers n and r, let $I^{(n+r)}(\mu)$ be the space of locally constant function $f\colon \widetilde{\mathrm{Sp}}_{r+k}(F) \to \mathbb{C}$ such that

$$f(\left(\begin{pmatrix} A & B \\ 0 & {}^t A^{-1} \end{pmatrix} \zeta\right) g) = \chi_{-1}^{1/2}(\det A, \zeta)^{n+r}\mu(\det A)|\det A|^{\frac{n+r+1}{2}} f(g).$$

It is called a degenerate principal series of $\widetilde{\mathrm{Sp}}_{n+r}(F)$. We consider an embedding

$$\mathrm{Sp}_n(F) \times \mathrm{Sp}_r(F) \hookrightarrow \mathrm{Sp}_{n+r}(F), \quad \left(\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\right) \mapsto \left(\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ \hline C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{array}\right).$$

Definition 2.1. *For an irreducible smooth representation π of $\widetilde{\mathrm{Sp}}_r(F)$ on which $\{\pm 1\}$ acts by $(\pm 1)^{n+r}$, the maximal π-isotypic quotient of $I^{(n+r)}(\mu)$ is of the form*

$$\mathcal{M}_\mu^{(n)}(\pi) \boxtimes \pi$$

for some smooth representation $\mathcal{M}_\mu^{(n)}(\pi)$ of $\widetilde{\mathrm{Sp}}_n(F)$. We call $\mathcal{M}_\mu^{(n)}(\pi)$ the (local) Miyawaki lift of π.

The following is basic properties of Miyawaki liftings.

Theorem 2.2 ([2, Theorems 1.8]). *Suppose that $n \ge r$.*

1. *For any π as above, $\mathcal{M}_\mu^{(n)}(\pi)$ is nonzero and of finite length.*
2. *If π is tempered, then $\mathcal{M}_\mu^{(n)}(\pi) \cong \mu\chi_{-1}^{[\frac{n+r}{2}]} \circ \det_{n-r} \rtimes \pi$.*
3. *Assume one of the following:*
   (a) π is discrete series;
   (b) π is tempered, and $r \le n \le r+1$ or $n > 2r$.

   Then all irreducible subquotients of $\mathcal{M}_\mu^{(r)}(\mathcal{M}_\mu^{(n)}(\pi))$ are isomorphic to π, and its maximal semisimple quotient is irreducible.

Miyawaki liftings satisfy seesaw identities.

Proposition 2.3 (Seesaw identity [2, Proposition 1.9]). *Let π and π' be irreducible representations of $\widetilde{\mathrm{Sp}}_{r-1}(F)$ and $\widetilde{\mathrm{Sp}}_n(F)$ on which $\{\pm 1\}$ acts by $(\pm 1)^{n+r}$ and $(\pm 1)^{n+r-1}$, respectively. Then*

$$\mathrm{Hom}_{\widetilde{J}_{r-1}(F)}(\mathcal{M}_{\mu}^{(r)}(\pi'), \pi \otimes \omega_{\psi_\xi}^{(r-1)}) \cong \mathrm{Hom}_{\widetilde{\mathrm{Sp}}_n(F)}(\mathcal{M}_{\mu\chi_\xi}^{(n)}(\pi) \otimes \omega_{\psi_\xi}^{(n)}, \pi').$$

We shall write these properties as the following seesaw diagram:

$$\begin{tikzcd}
\widetilde{\mathrm{Sp}}_r(F) \arrow[d, dash] \arrow[dr, dash] & \widetilde{\mathrm{Sp}}_n(F) \times \widetilde{\mathrm{Sp}}_n(F) \arrow[d, dash] \arrow[dl, dash] \\
\widetilde{\mathrm{Sp}}_{r-1}(F) \ltimes V_{r-1}(F) & \widetilde{\mathrm{Sp}}_n(F)
\end{tikzcd}.$$

References

[1] H. Atobe, *The local theta correspondence and the local Gan–Gross–Prasad conjecture for the symplectic-metaplectic case*, *Math. Ann.* (2017).

[2] H. Atobe, *A theory of Miyawaki liftings: The Hilbert-Siegel case*, arXiv:1712.03624v2.

[3] W. T. Gan, B. H. Gross and D. Prasad, *Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups*, *Astérisque*. No. **346** (2012), 1–109.

[4] B. Sun, *Multiplicity one theorems for Fourier-Jacobi models*, *Amer. J. Math.* **134** (2012), no. 6, 1655–1678.

Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914
JAPAN
E-mail address: `atobe@ms.u-tokyo.ac.jp`